Preintegrated IMU factor: Computation of the Jacobian Matrices

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Preliminaries

In this document we use the following first-order expressions for elements of the group SO(3):

$$\exp\left(\hat{\theta}+\tilde{\theta}\right) \approx \exp\left(\hat{\theta}\right)\exp\left(J_r(\hat{\theta})\tilde{\theta}\right) \approx \exp\left(\hat{\theta}\right)\left(\mathbf{I}+S\left(J_r(\hat{\theta})\tilde{\theta}\right)\right). \tag{1}$$

where is the right Jacobian of the exponential map on SO(3), see page 40 in [?], which can be written as:

$$J_r(\hat{\theta}) = \mathbf{I} - \frac{1-\cos(\|\hat{\theta}\|)}{\|\hat{\theta}\|^2} S\left(\hat{\theta}\right) + \frac{\|\hat{\theta}\|-\sin(\|\hat{\theta}\|)}{\|\hat{\theta}\|^3} S\left(\hat{\theta}\right)^2.$$

Another useful relation is the following:

$$\log\left(\exp\left(\hat{\theta}\right)\exp\left(\tilde{\theta}\right)\right) \approx \hat{\theta} + J_r^{-1}(\hat{\theta})\tilde{\theta} \tag{2}$$

where the inverse of the right Jacobian can be written explicitely as:

$$J_r^{-1}(\hat{\theta}) = \mathbf{I} + \frac{1}{2} S\left(\hat{\theta}\right) + \left(\frac{1}{\|\hat{\theta}\|^2} - \frac{1+\cos(\|\hat{\theta}\|)}{2\|\hat{\theta}\|\sin(\|\hat{\theta}\|)}\right) S\left(\hat{\theta}\right)^2.$$

Note that the Jacobian and its inverse reduce to the identity matrix for $\hat{\theta}=0$ (linearization around the identity).

1 Computation of the preintegrated measurements

1.1 Rotation

First-order approximation of the preintegrated rotation measurements as a function of the rotational bias:

$$
\begin{aligned}
R^1_{m+1} &= R^1_2 R^2_3 R^3_4 \dots R^m_{m+1} = \prod_{j=1}^{m} R^j_{j+1} \\
&= \hat{R}^1_2 \, \delta R^1_2 \, \hat{R}^2_3 \, \delta R^2_3 \dots \hat{R}^m_{m+1} \, \delta R^m_{m+1} \\
&\approx \hat{R}^1_2 \left(\mathbf{I} - \mathbf{S} \left(J^1_r \tilde{b}_\omega \Delta t_1 \right) \right) \hat{R}^2_3 \left(\mathbf{I} - \mathbf{S} \left(J^2_r \tilde{b}_\omega \Delta t_2 \right) \right) \dots \hat{R}^m_{m+1} \left(\mathbf{I} - \mathbf{S} \left(J^m_r \tilde{b}_\omega \Delta t_m \right) \right) \\
&= \left(\hat{R}^1_2 \hat{R}^2_3 - \hat{R}^1_2 \mathbf{S} \left(J^1_r \tilde{b}_\omega \Delta t_1 \right) \hat{R}^2_3 - \hat{R}^1_2 \hat{R}^2_3 \mathbf{S} \left(J^2_r \tilde{b}_\omega \Delta t_2 \right) \right) \dots \left(\hat{R}^m_{m+1} - \hat{R}^m_{m+1} \mathbf{S} \left(J^m_r \tilde{b}_\omega \Delta t_m \right) \right) \\
&= \hat{R}^1_2 \hat{R}^2_3 \dots \hat{R}^m_{m+1} - \sum_{i=1}^{m} \hat{R}^1_{i+1} \mathbf{S} \left(J^i_r \tilde{b}_\omega \Delta t_i \right) \hat{R}^{i+1}_{m+1} \\
&= \hat{R}^1_{m+1} - \sum_{i=1}^{m} \hat{R}^1_{i+1} \hat{R}^{i+1}_{m+1} \left(\hat{R}^{i+1}_{m+1} \right)^{-1} \mathbf{S} \left(J^i_r \tilde{b}_\omega \right) \hat{R}^{i+1}_{m+1} \Delta t_i \\
&= \hat{R}^1_{m+1} \left(\mathbf{I} - \sum_{i=1}^{m} \left(\hat{R}^{i+1}_{m+1} \right)^{-1} \mathbf{S} \left(J^i_r \tilde{b}_\omega \right) \hat{R}^{i+1}_{m+1} \Delta t_i \right) \\
&= \hat{R}^1_{m+1} \left(\mathbf{I} - \sum_{i=1}^{m} \mathbf{S} \left(\hat{R}^{i+1}_{m+1} J^i_r \tilde{b}_\omega \right) \Delta t_i \right) \\
&= \hat{R}^1_{m+1} \left(\mathbf{I} - \mathbf{S} \left(\sum_{i=1}^{m} \hat{R}^{i+1}_{m+1} J^i_r \tilde{b}_\omega \Delta t_i \right) \right)
\end{aligned} \tag{3}
$$

Ideally, the term $\sum_{i=1}^{m} \hat{R}^{i+1}_{m+1} J^i_r \Delta t_i$ would be recalculated every time the linearization point changes, but Lupton *et al.* suggest keeping it constant for efficiency.

When building the preintegrated measurements, we have the measured rotation rate $\bar{\omega}_j$ and current estimate of the bias $\hat{b}_\omega$, therefore, in the previous derivation, the linearization point of each rotation matrix is

$$
\hat{R}^j_{j+1} = \exp \left(\left(\bar{\omega}_j - \hat{b}_\omega \right) \Delta t_j \right).
$$

and

$$
J^j_r = J_r \left(\left(\bar{\omega}_j - \hat{b}_\omega \right) \Delta t_j \right).
$$

Later, as the bias changes, we do not recompute each $\hat{R}^j_{j+1}$, instead we recompute the preintegrated measurements using the first-order approximation in (3).

$$\acute{R}^1_{m+1}\left(\tilde{b}_\omega\right) = \hat{R}^1_{m+1}\exp\left(\frac{\partial R^1_{m+1}}{\partial b_\omega}\tilde{b}_\omega\right)$$
$$\overset{1}{\approx} \hat{R}^1_{m+1} + \hat{R}^1_{m+1}\mathbf{S}\left(\frac{\partial R^1_{m+1}}{\partial b_\omega}\tilde{b}_\omega\right), \text{ where}$$

$$\frac{\partial R^1_{m+1}}{\partial b_\omega} = -\sum_{i=1}^{m}\hat{R}^{i+1}_{m+1}J^i_r\Delta t_i = -\sum_{i=1}^{m-1}\hat{R}^{i+1}_{m+1}J^i_r\Delta t_i - J^m_r\Delta t_m = \hat{R}^{m+1}_m\frac{\partial R^1_m}{\partial b_\omega} - J^m_r\Delta t_m$$

1.2 Velocity

First-order approximation of the preintegrated velocity measurements as a function of the rotational and translational biases.

The accumulation of the velocity pseudo-measurement, of which we want the first-order approximation, is

$$\Delta v_{1m+1} = \sum_{j=1}^{m} R^1_j\left(\bar{a}^j - b_a\right)\Delta t_j$$

Splitting b_a into its linearization point $\hat{b}_a$ and its increment $\tilde{b}_a$, this is

$$\Delta v_{1m+1} = \sum_{j=1}^{m} R^1_j\left(\bar{a}^j - \hat{b}_a\right)\Delta t_j - \sum_{j=1}^{m} R^1_j\tilde{b}_a\Delta t_j$$

Substituting in the first-order approximation (3) for the relative rotation R^1_j,

$$\begin{aligned}
\Delta v_{1m+1} &\approx \sum_{j=1}^{m}\acute{R}^1_j\left(\tilde{b}_\omega\right)\left(\bar{a}^j - \hat{b}_a\right)\Delta t_j - \sum_{j=1}^{m}\acute{R}^1_N\left(\tilde{b}_\omega\right)\tilde{b}_a\Delta t_j \\
&\approx \sum_{j=1}^{m}\left(\hat{R}^1_j + \hat{R}^1_j\mathbf{S}\left(\frac{\partial R^1_j}{\partial b_\omega}\tilde{b}_\omega\right)\right)\left(\bar{a}^j - \hat{b}_a\right)\Delta t_j - \sum_{j=1}^{m}\hat{R}^1_j\tilde{b}_a\Delta t_j \;\; \text{[dropped 2nd-order terms]} \\
&= \sum_{j=1}^{m}\hat{R}^1_j\left(\bar{a}^j - \hat{b}_a\right)\Delta t_j + \sum_{j=1}^{m}\hat{R}^1_j\mathbf{S}\left(\frac{\partial R^1_j}{\partial b_\omega}\tilde{b}_\omega\right)\left(\bar{a}^j - \hat{b}_a\right)\Delta t_j - \left(\sum_{j=1}^{m}\hat{R}^1_j\Delta t_j\right)\tilde{b}_a \\
&= \sum_{j=1}^{m}\hat{R}^1_j\left(\bar{a}^j - \hat{b}_a\right)\Delta t_j - \left(\sum_{j=1}^{m}\hat{R}^1_j\mathbf{S}\left(\left(\bar{a}^j - \hat{b}_a\right)\Delta t_j\right)\frac{\partial R^1_j}{\partial b_\omega}\right)\tilde{b}_\omega - \left(\sum_{j=1}^{m}\hat{R}^1_j\Delta t_j\right)\tilde{b}_a \\
&= \Delta\hat{v}_{1m+1} + \frac{\partial\Delta v_{1m+1}}{\partial b_\omega}\tilde{b}_\omega + \frac{\partial\Delta v_{1m+1}}{\partial b_a}\tilde{b}_a
\end{aligned}$$

We derive recursive formulas for the constant term $\Delta\hat{v}_{1m+1}$ and Jacobians

$\partial \Delta v_{1m+1} / \partial b_\omega$ and $\partial \Delta v_{1m+1} / \partial b_a$,

$$\begin{aligned}
\Delta \hat{v}_{1m+1} &= \sum_{j=1}^{m} \hat{R}_j^1 \left(\bar{a}^j - \hat{b}_a \right) \Delta t_j \\
&= \sum_{j=1}^{m-1} \hat{R}_j^1 \left(\bar{a}^j - \hat{b}_a \right) \Delta t_j + \hat{R}_m^1 \left(\bar{a}^m - \hat{b}_a \right) \Delta t_m \\
&= \Delta \hat{v}_{1m} + \hat{R}_m^1 \left(\bar{a}^m - \hat{b}_a \right) \Delta t_m
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \Delta v_{1m+1}}{\partial b_\omega} &= -\sum_{j=1}^{m} \hat{R}_j^1 \mathbf{S} \left(\left(\bar{a}^j - \hat{b}_a \right) \Delta t_j \right) \frac{\partial R_j^1}{\partial b_\omega} \\
&= -\sum_{j=1}^{m-1} \hat{R}_j^1 \mathbf{S} \left(\left(\bar{a}^j - \hat{b}_a \right) \Delta t_j \right) \frac{\partial R_j^1}{\partial b_\omega} - \hat{R}_m^1 \mathbf{S} \left(\left(\bar{a}^m - \hat{b}_a \right) \Delta t_m \right) \frac{\partial R_m^1}{\partial b_\omega} \\
&= \frac{\partial \Delta v_{1m}}{\partial b_\omega} - \hat{R}_m^1 \mathbf{S} \left(\left(\bar{a}^m - \hat{b}_a \right) \Delta t_m \right) \frac{\partial R_m^1}{\partial b_\omega}
\end{aligned}$$

$$\frac{\partial \Delta v_{1m+1}}{\partial b_a} = -\sum_{j=1}^{m} \hat{R}_j^1 \Delta t_j = -\sum_{j=1}^{m-1} \hat{R}_j^1 \Delta t_j - \hat{R}_m^1 \Delta t_m = \frac{\partial \Delta v_{1m}}{\partial b_a} - \hat{R}_m^1 \Delta t_m$$

1.3 Position

First-order approximation of the preintegrated position measurements as a function of the rotational and translational biases.

The accumulation of the position pseudo-measurement, of which we want the first-order approximation, is

$$\Delta p_{1m+1} = \sum_{k=1}^{m} \left(\Delta v_{1k} \Delta t_k + \frac{1}{2} R_k^1 \left(\bar{a}^k - b_a \right) \Delta t_k^2 \right)$$

Splitting b_a into its linearization point $\hat{b}_a$ and its increment $\tilde{b}_a$, this is

$$\Delta p_{1m+1} = \sum_{k=1}^{m} \Delta v_{1k} \Delta t_k + \frac{1}{2} R_k^1 \left(\bar{a}^k - \hat{b}_a \right) \Delta t_k^2 - \frac{1}{2} R_k^1 \tilde{b}_a \Delta t_k^2$$

Substituting the first-order approximation of Δv_{1k} and of the rotation R_k^1,

$$\begin{aligned}
\Delta p_{1m+1} \quad = \quad & \sum_{k=1}^{m} \left(\Delta \hat{v}_{1k} + \frac{\partial \Delta v_{1k}}{\partial b_\omega} \tilde{b}_\omega + \frac{\partial \Delta v_{1k}}{\partial b_a} \tilde{b}_a \right) \Delta t_k \\
& + \frac{1}{2} \sum_{k=1}^{m} \acute{R}_k^1 \left(\tilde{b}_\omega \right) \left(\bar{a}^k - \hat{b}_a \right) \Delta t_k^2 - \frac{1}{2} \sum_{k=1}^{m} \acute{R}_k^1 \left(\tilde{b}_\omega \right) \tilde{b}_a \Delta t_k^2
\end{aligned}$$

Expanding $\acute{R}_k^1\left(\tilde{b}_\omega\right)$, we have

$$\begin{aligned}
\Delta p_{1m+1} &= \sum_{k=1}^{m}\left(\Delta\hat{v}_{1k} + \frac{\partial \Delta v_{1k}}{\partial b_\omega}\tilde{b}_\omega + \frac{\partial \Delta v_{1k}}{\partial b_a}\tilde{b}_a\right)\Delta t_k \\
&+\frac{1}{2}\sum_{k=1}^{m}\left(\hat{R}_k^1 + \hat{R}_k^1\mathbf{S}\left(\frac{\partial R_k^1}{\partial b_\omega}\tilde{b}_\omega\right)\right)\left(\bar{a}^k - \hat{b}_a\right)\Delta t_k^2 \\
&-\frac{1}{2}\sum_{k=1}^{m}\left(\hat{R}_k^1 + \hat{R}_k^1\mathbf{S}\left(\frac{\partial R_k^1}{\partial b_\omega}\tilde{b}_\omega\right)\right)\tilde{b}_a\Delta t_k^2
\end{aligned}$$

Dropping second-order terms,

$$\begin{aligned}
\Delta p_{1m+1} &= \sum_{k=1}^{m}\left(\Delta\hat{v}_{1k} + \frac{\partial \Delta v_{1k}}{\partial b_\omega}\tilde{b}_\omega + \frac{\partial \Delta v_{1k}}{\partial b_a}\tilde{b}_a\right)\Delta t_k \\
&+\frac{1}{2}\sum_{k=1}^{m}\left(\hat{R}_k^1 + \hat{R}_k^1\mathbf{S}\left(\frac{\partial R_k^1}{\partial b_\omega}\tilde{b}_\omega\right)\right)\left(\bar{a}^k - \hat{b}_a\right)\Delta t_k^2 - \frac{1}{2}\sum_{k=1}^{m}\hat{R}_k^1\tilde{b}_a\Delta t_k^2
\end{aligned}$$

Rearranging,

$$\begin{aligned}
\Delta p_{1m+1} &= \sum_{k=1}^{m}\left(\Delta\hat{v}_{1k}\Delta t_k + \frac{1}{2}\hat{R}_k^1\left(\bar{a}^k - \hat{b}_a\right)\Delta t_k^2\right) \\
&+\sum_{k=1}^{m}\left(\frac{\partial \Delta v_{1k}}{\partial b_\omega}\tilde{b}_\omega + \frac{\partial \Delta v_{1k}}{\partial b_a}\tilde{b}_a\right)\Delta t_k \\
&+\frac{1}{2}\sum_{k=1}^{m}\hat{R}_k^1\mathbf{S}\left(\frac{\partial R_k^1}{\partial b_\omega}\tilde{b}_\omega\right)\left(\bar{a}^k - \hat{b}_a\right)\Delta t_k^2 - \frac{1}{2}\sum_{k=1}^{m}\hat{R}_k^1\tilde{b}_a\Delta t_k^2
\end{aligned}$$

recalling that for vectors a and b it holds $\mathbf{S}(a)b = -\mathbf{S}(b)a$:

$$\begin{aligned}
\Delta p_{1m+1} &= \sum_{k=1}^{m}\left(\Delta\hat{v}_{1k}\Delta t_k + \frac{1}{2}\hat{R}_k^1\left(\bar{a}^k - \hat{b}_a\right)\Delta t_k^2\right) \\
&+\sum_{k=1}^{m}\left(\frac{\partial \Delta v_{1k1}}{\partial b_\omega}\tilde{b}_\omega + \frac{\partial \Delta v_{1k}}{\partial b_a}\tilde{b}_a\right)\Delta t_k \\
&-\frac{1}{2}\sum_{k=1}^{m}\hat{R}_k^1\mathbf{S}\left(\bar{a}^k - \hat{b}_a\right)\Delta t_k^2\frac{\partial R_k^1}{\partial b_\omega}\tilde{b}_\omega - \frac{1}{2}\sum_{k=1}^{m}\hat{R}_k^1\tilde{b}_a\Delta t_k^2
\end{aligned}$$

rearranging the terms:

$$\begin{aligned}\Delta p_{1m+1} \quad = \quad & \sum_{k=1}^{m}\left(\Delta\hat{v}_{1k}\Delta t_k + \frac{1}{2}\hat{R}_k^1\left(\bar{a}^k - \hat{b}_a\right)\Delta t_k^2\right) \\ & + \sum_{k=1}^{m}\left(\frac{\partial \Delta v_{1k}}{\partial b_\omega}\Delta t_k - \frac{1}{2}\hat{R}_k^1\mathbf{S}\left(\bar{a}^k - \hat{b}_a\right)\Delta t_k^2\frac{\partial R_k^1}{\partial b_\omega}\right)\tilde{b}_\omega \\ & + \sum_{k=1}^{m}\left(\frac{\partial \Delta v_{1k}}{\partial b_a}\Delta t_k - \frac{1}{2}\hat{R}_k^1\Delta t_k^2\right)\tilde{b}_a = \Delta\hat{p}_{1m+1} + \frac{\partial \Delta p_{1m+1}}{\partial b_\omega}\tilde{b}_\omega + \frac{\partial \Delta p_{1m+1}}{\partial b_a}\tilde{b}_a\end{aligned}$$

We derive recursive formulas for the constant term $\Delta\hat{p}_{1m+1}$ and Jacobians $\partial\Delta p_{1m+1}/\partial b_\omega$ and $\partial\Delta p_{1m+1}/\partial b_a$,

$$\begin{aligned}\Delta\hat{p}_{1m+1} &= \sum_{k=1}^{m}\left(\Delta\hat{v}_{1k}\Delta t_k + \frac{1}{2}\hat{R}_k^1\left(\bar{a}^k - \hat{b}_a\right)\Delta t_k^2\right) \\ &= \sum_{k=1}^{m-1}\left(\Delta\hat{v}_{1k}\Delta t_k + \frac{1}{2}\hat{R}_k^1\left(\bar{a}^k - \hat{b}_a\right)\Delta t_k^2\right) \\ &+ \Delta\hat{v}_{1m}\Delta t_m + \frac{1}{2}\hat{R}_m^1\left(\bar{a}^m - \hat{b}_a\right)\Delta t_m^2 \\ &= \Delta\hat{p}_{1m} + \left(\Delta\hat{v}_{1m}\Delta t_m + \frac{1}{2}\hat{R}_m^1\left(\bar{a}^m - \hat{b}_a\right)\Delta t_m^2\right)\end{aligned}$$

$$\begin{aligned}\frac{\partial \Delta p_{1m+1}}{\partial b_\omega} &= \sum_{k=1}^{m}\left(\frac{\partial \Delta v_{1k}}{\partial b_\omega}\Delta t_k - \frac{1}{2}\hat{R}_k^1\mathbf{S}\left(\bar{a}^k - \hat{b}_a\right)\Delta t_k^2\frac{\partial R_k^1}{\partial b_\omega}\right) \\ &= \sum_{k=1}^{m-1}\left(\frac{\partial \Delta v_{1k}}{\partial b_\omega}\Delta t_k - \frac{1}{2}\hat{R}_k^1\mathbf{S}\left(\bar{a}^k - \hat{b}_a\right)\Delta t_k^2\frac{\partial R_k^1}{\partial b_\omega}\right) \\ &+ \left(\frac{\partial \Delta v_{1m}}{\partial b_\omega}\Delta t_m - \frac{1}{2}\hat{R}_m^1\mathbf{S}\left(\bar{a}^m - \hat{b}_a\right)\Delta t_m^2\frac{\partial R_m^1}{\partial b_\omega}\right) \\ &= \frac{\partial \Delta p_{1m}}{\partial b_\omega} + \left(\frac{\partial \Delta v_{1m}}{\partial b_\omega}\Delta t_m - \frac{1}{2}\hat{R}_m^1\mathbf{S}\left(\bar{a}^m - \hat{b}_a\right)\Delta t_m^2\frac{\partial R_m^1}{\partial b_\omega}\right)\end{aligned}$$

$$\begin{aligned}\frac{\partial \Delta p_{1m+1}}{\partial b_a} &= \sum_{k=1}^{m}\left(\frac{\partial \Delta v_{1k}}{\partial b_a}\Delta t_k - \frac{1}{2}\hat{R}_k^1\tilde{b}_a\Delta t_k^2\right) \\ &= \sum_{k=1}^{m-1}\left(\frac{\partial \Delta v_{1k}}{\partial b_a}\Delta t_k - \frac{1}{2}\hat{R}_k^1\tilde{b}_a\Delta t_k^2\right) + \left(\frac{\partial \Delta v_{1m}}{\partial b_a}\Delta t_m - \frac{1}{2}\hat{R}_m^1\Delta t_m^2\right) \\ &= \frac{\partial \Delta p_{1m}}{\partial b_a} + \left(\frac{\partial \Delta v_{1m}}{\partial b_a}\Delta t_m - \frac{1}{2}\hat{R}_m^1\Delta t_m^2\right)\end{aligned}$$

Some final comments:

1. the number of measurements we are integrating is m (starting from pose 1 and integrating over m intervals we reach pose $m+1$)

2. All the previous expressions use the following initial conditions: $R_1^1 = \mathbf{I}$, $\Delta p_{11} = 0$, and $\Delta v_{11} = 0$. Similar initial conditions hold for the corresponding "hat" quantities.

3. Similarly, for the Jacobian, one has to consider the following initial conditions: $\partial \Delta v_{11}/\partial b_\omega = 0$, $\partial \Delta v_{11}/\partial b_a = 0$, $\partial R_1^1/\partial b_\omega = 0$, $\partial \Delta v_{11}/\partial b_\omega = 0$, $\partial \Delta v_{11}/\partial b_a = 0$.

2 Factor Modelling and Jacobian

In the previuos section we showed how to summarize m measurements so to produce a single relative measurement (position, velocity, rotation) between frame 1 and frame N. We now tailor the formulation to the case in which the measurements are integrated between two consecutive robot poses, namely, i and j. Therefore, we rename the measurements as follows, dropping the dependence on the number of integrated measurements $(N-1)$:

$$
\begin{aligned}
&\Delta p_{ij} = \Delta p_{1m+1} \\
&\Delta v_{ij} = \Delta p_{1m+1} \\
&R_j^i = R_{m+1}^1
\end{aligned}
$$

We start writing the most general expression for the IMU factors:

$$
\begin{aligned}
f_p &= p_j^{L_j} - R_{L_i}^{L_j}\left(p_i^{L_i} + R_i^{L_i}\Delta p_{ij} + v_i^{L_i}\Delta t + \left(\frac{1}{2}g^{L_i} - \mathbf{S}(\omega_{iL_i}^{L_i})v_i^{L_i}\right)\Delta t^2\right) \\
f_v &= v_j^{L_j} - R_{L_i}^{L_j}\left(v_i^{L_i} + R_i^{L_i}\Delta v_{ij} + \left(g^{L_i} - 2\mathbf{S}(\omega_{iL_i}^{L_i})v_i^{L_i}\right)\Delta t\right) \\
f_R &= \left(R_{L_i}^{L_j}\left(R_i^{L_i} exp\Delta\phi\right)\right)^{-1} R_j^{L_j}, \quad with \quad \Delta\phi = log(R_j^i) - R_{L_i}^i \omega_{iL_i}^{L_i}\Delta t
\end{aligned}
$$

The following sections derive the expressions of the Jacobians, considering different models (sorted by increasing complexity) in which we consider different sets of assumptions (e.g., null bias, or negligible Coriolis).

2.1 Model 1: neglecting biases and Coriolis effect

Assumptions:

1. $R_{L_i}^{L_j} = \mathbf{I}$
2. $b_a = 0$ and $b_\omega = 0$
3. $\omega_{iL_i}^{L_i} = 0$

4. known initial conditions $(g^{L_i}, p_0^{L_0}, v_0^{L_0}, R_0^{L_0})$

We first recall the simplified expressions for the factors:

$$
\begin{aligned}
f_p &= p_j^{L_i} - p_i^{L_i} - R_i^{L_i}\Delta p_{ij} - v_i^{L_i}\Delta t - \frac{1}{2} g^{L_i}\Delta t^2 \\
f_v &= v_j^{L_i} - v_i^{L_i} - R_i^{L_i}\Delta v_{ij} - g^{L_i}\Delta t \\
f_R &= \left(R_j^i\right)^{-1}\left(R_i^{L_i}\right)^{-1} R_j^{L_i}
\end{aligned}
$$

Then we proceed in the linearization of the factors.

2.1.1 Position

$$
\begin{aligned}
f_p &= p_j^{L_i} - p_i^{L_i} - R_i^{L_i}\Delta p_{ij} - v_i^{L_i}\Delta t - \frac{1}{2} g^{L_i}\Delta t^2 \\
&\approx \left(\hat{p}_j^{L_i} - \hat{p}_i^{L_i} - \hat{R}_i^{L_i}\Delta p_{ij} - \hat{v}_i^{L_i}\Delta t - \frac{1}{2} g^{L_i}\Delta t^2\right) + J_p\delta x
\end{aligned}
$$

where

$$
\begin{array}{rcccccccccccc}
J_p & = & [0 & 0 & 0 & \dots & -\hat{R}_i^{L_i} & \hat{R}_i^{L_i}\mathbf{S}(\Delta p_{ij}) & -\mathbf{I}\boldsymbol{\Delta} t & \hat{R}_j^{L_i} 0 & 0 & 0] \\
\delta x & = & [\delta p_1 & \delta\theta_1 & \delta v_1 & \dots & \delta p_i & \delta\theta_i & \delta v_i & \delta p_j & \delta\theta_j & \delta v_j]^T
\end{array}
$$

2.1.2 Velocity

$$
\begin{aligned}
f_v &= v_j^{L_i} - v_i^{L_i} - R_i^{L_i}\Delta v_{ij} - g^{L_i}\Delta t \\
&\approx \left(\hat{v}_j^{L_i} - \hat{v}_i^{L_i} - \hat{R}_i^{L_i}\Delta v_{ij} - g^{L_i}\Delta t\right) + J_v\delta x
\end{aligned}
$$

where

$$
\begin{array}{rcccccccccccc}
J_v & = & [0 & 0 & 0 & \dots & 0 & \hat{R}_i^{L_i}\mathbf{S}(\Delta v_{ij}) & -\mathbf{I} & 0 & 0 & \mathbf{I}] \\
\delta x & = & [\delta p_1 & \delta\theta_1 & \delta v_1 & \dots & \delta p_i & \delta\theta_i & \delta v_i & \delta p_j & \delta\theta_j & \delta v_j]^T
\end{array}
$$

2.1.3 Rotation

$$
\begin{aligned}
f_R &= \left(R_j^i\right)^{-1}\left(R_i^{L_i}\right)^{-1} R_j^{L_i} \\
&\approx \left(\left(\hat{R}_j^i\right)^{-1}\left(\hat{R}_i^{L_i}\right)^{-1}\hat{R}_j^{L_i}\right)\left(\mathbf{I} + \mathbf{S}\left(\delta\theta_j - \left(\hat{R}_j^{L_i}\right)^{-1}\hat{R}_i^{L_i}\delta\theta_i\right)\right)
\end{aligned}
$$

where

$$
\begin{array}{rcl}
J_R & = & [0 \quad 0 \quad 0 \quad \dots \quad 0 \quad -\left(\hat{R}_j^{L_i}\right)^{-1}\hat{R}_i^{L_i} \quad 0 \quad 0 \quad \mathbf{I} \quad 0] \\
\delta x & = & [\delta p_1 \quad \delta\theta_1 \quad \delta v_1 \quad \dots \quad \delta p_i \quad \delta\theta_i \quad \delta v_i \quad \delta p_j \quad \delta\theta_j \quad \delta v_j]^T
\end{array}
$$

2.2 Model 2: neglecting Coriolis effect

Assumptions:

1. $R_{L_i}^{L_j} = \mathbf{I}$
2. $\omega_{iL_i}^{L_i} = 0$
3. known initial conditions $(g^{L_i}, p_0^{L_0}, v_0^{L_0}, R_0^{L_0})$

We first recall the simplified expressions for the factors, approximating a first-order dependence of
the preintegrated measurements on the biases:

$$
\begin{array}{rcl}
f_p & = & p_j^{L_i} - p_i^{L_i} - R_i^{L_i}\left(\Delta\hat{p}_{ij} + \frac{\partial\Delta p_{ij}}{\partial b_\omega}\tilde{b}_\omega + \frac{\partial\Delta p_{ij}}{\partial b_a}\tilde{b}_a\right) - v_i^{L_i}\Delta t - \frac{1}{2}g^{L_i}\Delta t^2 \\
f_v & = & v_j^{L_i} - v_i^{L_i} - R_i^{L_i}\left(\Delta\hat{v}_{ij} + \frac{\partial\Delta v_{ij}}{\partial b_\omega}\tilde{b}_\omega + \frac{\partial\Delta v_{ij}}{\partial b_a}\tilde{b}_a\right) - g^{L_i}\Delta t \\
f_R & = & \left(\hat{R}_j^i \exp\left(\frac{\partial R_j^i}{\partial b_\omega}\tilde{b}_\omega\right)\right)^{-1}\left(R_i^{L_i}\right)^{-1} R_j^{L_i}
\end{array}
$$

Then we proceed in the linearization of the factors.

2.2.1 Position

$$
f_p = p_j^{L_i} - p_i^{L_i} - R_i^{L_i}\Delta\hat{p}_{ij} - R_i^{L_i}\frac{\partial\Delta p_{ij}}{\partial b_\omega}\tilde{b}_\omega - R_i^{L_i}\frac{\partial\Delta p_{ij}}{\partial b_a}\tilde{b}_a - v_i^{L_i}\Delta t - \frac{1}{2}g^{L_i}\Delta t^2
$$

It still remain to linearize the terms that depend on the biases (since they also depend nonlinearly on the rotation matrix $R_i^{L_i}$). For this purpose we rewrite

$\tilde{b}_\omega = \hat{\tilde{b}}_\omega + \delta\tilde{b}_\omega$ and $\tilde{b}_a = \hat{\tilde{b}}_a + \delta\tilde{b}_a$.

$$
\begin{aligned}
f_p &\approx p_j^{L_i} - p_i^{L_i} - R_i^{L_i}\Delta\hat{p}_{ij} - \hat{R}_i^{L_i}\left(\mathbf{I} + \mathbf{S}(\delta\theta_i)\right)\frac{\partial\Delta p_{ij}}{\partial b_\omega}\left(\hat{\tilde{b}}_\omega + \delta\tilde{b}_\omega\right) \\
&\quad -\hat{R}_i^{L_i}\left(\mathbf{I} + \mathbf{S}(\delta\theta_i)\right)\frac{\partial\Delta p_{ij}}{\partial b_a}\left(\hat{\tilde{b}}_a + \delta\tilde{b}_a\right) - v_i^{L_i}\Delta t - \frac{1}{2}g^{L_i}\Delta t^2 \\
&\approx p_j^{L_i} - p_i^{L_i} - R_i^{L_i}\Delta\hat{p}_{ij} - \hat{R}_i^{L_i}\frac{\partial\Delta p_{ij}}{\partial b_\omega}\left(\hat{\tilde{b}}_\omega\right) - \hat{R}_i^{L_i}\mathbf{S}\left(\frac{\partial\Delta p_{ij}}{\partial b_\omega}\hat{\tilde{b}}_\omega\right)\delta\theta_i - \hat{R}_i^{L_i}\frac{\partial\Delta p_{ij}}{\partial b_\omega}\left(\delta\tilde{b}_\omega\right) \\
&\quad -\hat{R}_i^{L_i}\frac{\partial\Delta p_{ij}}{\partial b_a}\left(\hat{\tilde{b}}_a\right) - \hat{R}_i^{L_i}S\left(\frac{\partial\Delta p_{ij}}{\partial b_a}\hat{\tilde{b}}_a\right)\delta\theta_i - \hat{R}_i^{L_i}\frac{\partial\Delta p_{ij}}{\partial b_a}\left(\delta\tilde{b}_a\right) - v_i^{L_i}\Delta t - \frac{1}{2}g^{L_i}\Delta t^2 =
\end{aligned}
$$

$$
f_p \approx \left(\hat{p}_j^{L_i} - \hat{p}_i^{L_i} - \hat{R}_i^{L_i}\Delta\hat{p}_{ij} - \hat{R}_i^{L_i}\frac{\partial\Delta p_{ij}}{\partial b_\omega}\left(\hat{\tilde{b}}_\omega\right) - \hat{R}_i^{L_i}\frac{\partial\Delta p_{ij}}{\partial b_a}\left(\hat{\tilde{b}}_a\right) - \hat{v}_i^{L_i}\Delta t - \frac{1}{2}g^{L_i}\Delta t^2\right) + J_p\delta x
$$

with

$$
\begin{aligned}
J_p &= [-\hat{R}_i^{L_i} \quad J_{\theta_i}^p \quad -\mathbf{I\Delta}t \quad -\hat{R}_i^{L_i}\tfrac{\partial\Delta p_{ij}}{\partial b_a} \quad -\hat{R}_i^{L_i}\tfrac{\partial\Delta p_{ij}}{\partial b_\omega} \quad \hat{R}_j^{L_i} \quad 0 \quad 0] \\
\delta x &= [\delta p_i \quad \delta\theta_i \quad \delta v_i \quad \delta\tilde{b}_{ai} \quad \delta\tilde{b}_{\omega i} \quad \delta p_j \quad \delta\theta_j \quad \delta v_j]^T
\end{aligned}
$$

and $J_{\theta_i}^p = \hat{R}_i^{L_i}\mathbf{S}\left(\Delta\hat{p}_{ij} + \frac{\partial\Delta p_{ij}}{\partial b_\omega}\hat{\tilde{b}}_\omega + \frac{\partial\Delta p_{ij}}{\partial b_a}\hat{\tilde{b}}_a\right)$.

2.2.2 Velocity

$$
\begin{aligned}
f_v &= v_j^{L_i} - v_i^{L_i} - R_i^{L_i}\left(\Delta\hat{v}_{ij} + \frac{\partial\Delta v_{ij}}{\partial b_\omega}\tilde{b}_\omega + \frac{\partial\Delta v_{ij}}{\partial b_a}\tilde{b}_a\right) - g^{L_i}\Delta t \\
&\approx v_j^{L_i} - v_i^{L_i} - R_i^{L_i}\Delta\hat{v}_{ij} - g^{L_i}\Delta t - R_i^{L_i}\frac{\partial\Delta v_{ij}}{\partial b_\omega}\tilde{b}_\omega - R_i^{L_i}\frac{\partial\Delta v_{ij}}{\partial b_a}\tilde{b}_a
\end{aligned}
$$

It still remain to linearize the terms that depend on the biases (since they also depend nonlinearly on the rotation matrix $R_i^{L_i}$). For this purpose we rewrite $\tilde{b}_\omega = \hat{\tilde{b}}_\omega + \delta\tilde{b}_\omega$ and $\tilde{b}_a = \hat{\tilde{b}}_a + \delta\tilde{b}_a$.

$$
\begin{aligned}
f_v &= v_j^{L_i} - v_i^{L_i} - R_i^{L_i}\Delta\hat{v}_{ij} - g^{L_i}\Delta t - R_i^{L_i}\frac{\partial \Delta v_{ij}}{\partial b_\omega}\tilde{b}_\omega - R_i^{L_i}\frac{\partial \Delta v_{ij}}{\partial b_a}\tilde{b}_a \\
&\approx v_j^{L_i} - v_i^{L_i} - R_i^{L_i}\Delta\hat{v}_{ij} - g^{L_i}\Delta t - \hat{R}_i^{L_i}\left(\mathbf{I}+\mathbf{S}(\delta\theta_i)\right)\frac{\partial \Delta v_{ij}}{\partial b_\omega}\left(\hat{\tilde{b}}_\omega + \delta\tilde{b}_\omega\right) \\
&\quad -\hat{R}_i^{L_i}\left(\mathbf{I}+\mathbf{S}(\delta\theta_i)\right)\frac{\partial \Delta v_{ij}}{\partial b_a}\left(\hat{\tilde{b}}_a + \delta\tilde{b}_a\right) = \\
&\approx v_j^{L_i} - v_i^{L_i} - R_i^{L_i}\Delta\hat{v}_{ij} - g^{L_i}\Delta t - \hat{R}_i^{L_i}\frac{\partial \Delta v_{ij}}{\partial b_\omega}\hat{\tilde{b}}_\omega \\
&\quad -\hat{R}_i^{L_i}\mathbf{S}(\delta\theta_i)\frac{\partial \Delta v_{ij}}{\partial b_\omega}\hat{\tilde{b}}_\omega - \hat{R}_i^{L_i}\frac{\partial \Delta v_{ij}}{\partial b_\omega}\delta\tilde{b}_\omega - \hat{R}_i^{L_i}\frac{\partial \Delta v_{ij}}{\partial b_a}\hat{\tilde{b}}_a \\
&\quad -\hat{R}_i^{L_i}\mathbf{S}(\delta\theta_i)\frac{\partial \Delta v_{ij}}{\partial b_a}\hat{\tilde{b}}_a - \hat{R}_i^{L_i}\frac{\partial \Delta v_{ij}}{\partial b_a}\delta\tilde{b}_a = \\
&= v_j^{L_i} - v_i^{L_i} - R_i^{L_i}\Delta\hat{v}_{ij} - g^{L_i}\Delta t - \hat{R}_i^{L_i}\frac{\partial \Delta v_{ij}}{\partial b_\omega}\hat{\tilde{b}}_\omega - \hat{R}_i^{L_i}\frac{\partial \Delta v_{ij}}{\partial b_a}\hat{\tilde{b}}_a \\
&\quad +\hat{R}_i^{L_i}\mathbf{S}\left(\frac{\partial \Delta v_{ij}}{\partial b_\omega}\hat{\tilde{b}}_\omega\right)\delta\theta_i - \hat{R}_i^{L_i}\frac{\partial \Delta v_{ij}}{\partial b_\omega}\delta\tilde{b}_\omega + \hat{R}_i^{L_i}\mathbf{S}\left(\frac{\partial \Delta v_{ij}}{\partial b_a}\hat{\tilde{b}}_a\right)\delta\theta_i - \hat{R}_i^{L_i}\frac{\partial \Delta v_{ij}}{\partial b_a}\delta\tilde{b}_a \\
&= \left(\hat{v}_j^{L_i} - \hat{v}_i^{L_i} - \hat{R}_i^{L_i}\Delta\hat{v}_{ij} - \hat{R}_i^{L_i}\frac{\partial \Delta v_{ij}}{\partial b_\omega}\hat{\tilde{b}}_\omega - \hat{R}_i^{L_i}\frac{\partial \Delta v_{ij}}{\partial b_a}\hat{\tilde{b}}_a - g^{L_i}\Delta t\right) + J_v\delta x
\end{aligned}
$$

where

$$
\begin{array}{rcllllllll}
J_p &=& [0 & J_{\theta_i}^v - \mathbf{I} & -\hat{R}_i^{L_i}\frac{\partial \Delta v_{ij}}{\partial b_a} & -\hat{R}_i^{L_i}\frac{\partial \Delta v_{ij}}{\partial b_\omega} & 0 & 0 & \mathbf{I}] \\
\delta x &=& [\delta p_i & \delta\theta_i & \delta v_i & \delta\tilde{b}_{ai} & \delta\tilde{b}_{\omega i} & \delta p_j & \delta\theta_j & \delta v_j]^T
\end{array}
$$

and $J_{\theta_i}^v = \hat{R}_i^{L_i}\mathbf{S}\left(\Delta\hat{v}_{ij} + \frac{\partial \Delta v_{ij}}{\partial b_\omega}\hat{\tilde{b}}_\omega + \frac{\partial \Delta v_{ij}}{\partial b_a}\hat{\tilde{b}}_a\right)$.

2.2.3 Rotation

$$
\begin{aligned}
f_R &= \left(\hat{R}_j^i \exp\left(\frac{\partial R_j^i}{\partial b_\omega}\tilde{b}_\omega\right)\right)^{-1}\left(R_i^{L_i}\right)^{-1} R_j^{L_i} \approx \\
&\approx \left(\hat{R}_j^i \exp\left(\frac{\partial R_j^i}{\partial b_\omega}\hat{\tilde{b}}_\omega + \frac{\partial R_j^i}{\partial b_\omega}\delta\tilde{b}_\omega\right)\right)^{-1}\left(\hat{R}_i^{L_i}\delta R_i^{L_i}\right)^{-1}\left(\hat{R}_j^{L_i}\delta R_j^{L_i}\right) = \\
&\approx \left[\hat{R}_j^i \exp\left(\theta_{BIAS}\right)\left(\mathbf{I}+\mathbf{S}\left(J_r\left(\theta_{BIAS}\right)\frac{\partial R_j^i}{\partial b_\omega}\delta\tilde{b}_\omega\right)\right)\right]^{-1} \\
&\quad \left(\hat{R}_i^{L_i}\left(\mathbf{I}+\mathbf{S}\left(\delta\theta_i\right)\right)\right)^{-1}\left(\hat{R}_j^{L_i}\left(\mathbf{I}+\mathbf{S}\left(\delta\theta_j\right)\right)\right) = \\
&= \left[\hat{R}_{BC}\left(\mathbf{I}+\mathbf{S}\left(J_r(\theta_{BIAS})\frac{\partial R_j^i}{\partial b_\omega}\delta\tilde{b}_\omega\right)\right)\right]^{-1}\left(\hat{R}_i^{L_i}\left(\mathbf{I}+\mathbf{S}\left(\delta\theta_i\right)\right)\right)^{-1}\left(\hat{R}_j^{L_i}\left(\mathbf{I}+\mathbf{S}\left(\delta\theta_j\right)\right)\right)
\end{aligned}
$$

with

$$\hat{R}_{BC} = \hat{R}_j^i \exp\left(\frac{\partial R_j^i}{\partial b_\omega}\hat{\tilde{b}}_\omega\right) \qquad \theta_{BIAS} = \frac{\partial R_j^i}{\partial b_\omega}\hat{\tilde{b}}_\omega$$

In the previous expression we used the small angle approximations for small (error) rotations and we applyed equation (1) for the first order approximation of the exponential map containing biases. Developing the expression we obtian:

$$\begin{aligned}
f_R \;&\approx\; \left(\mathbf{I} - \mathbf{S}\left(J_r(\theta_{BIAS})\frac{\partial R_j^i}{\partial b_\omega}\delta\tilde{b}_\omega\right)\right)\hat{R}_{BC}^{-1}\left(\mathbf{I} - \mathbf{S}\left(\delta\theta_i\right)\right)\left(\hat{R}_i^{L_i}\right)^{-1}\left(\hat{R}_j^{L_i}\left(\mathbf{I} + \mathbf{S}\left(\delta\theta_j\right)\right)\right) = \\
&= \left(\mathbf{I} - \mathbf{S}\left(J_r(\theta_{BIAS})\frac{\partial R_j^i}{\partial b_\omega}\delta\tilde{b}_\omega\right)\right) \\
&\quad \left(\hat{R}_{BC}^{-1}\left(\hat{R}_i^{L_i}\right)^{-1}\hat{R}_j^{L_i} - \hat{R}_{BC}^{-1}\mathbf{S}\left(\delta\theta_i\right)\left(\hat{R}_i^{L_i}\right)^{-1}\hat{R}_j^{L_i} + \hat{R}_{BC}^{-1}\left(\hat{R}_i^{L_i}\right)^{-1}\hat{R}_j^{L_i}\mathbf{S}\left(\delta\theta_j\right)\right) = \\
&= \left(\mathbf{I} - \mathbf{S}\left(J_r(\theta_{BIAS})\frac{\partial R_j^i}{\partial b_\omega}\delta\tilde{b}_\omega\right)\right) \\
&\quad \left(\hat{f}_R - \hat{R}_{BC}^{-1}\left(\hat{R}_i^{L_i}\right)^{-1}\hat{R}_j^{L_i}\mathbf{S}\left(\left(\left(\hat{R}_i^{L_i}\right)^{-1}\hat{R}_j^{L_i}\right)^{-1}\delta\theta_i\right) + \hat{f}_R\mathbf{S}\left(\delta\theta_j\right)\right) = \\
&= \left(\mathbf{I} - \mathbf{S}\left(J_r(\theta_{BIAS})\frac{\partial R_j^i}{\partial b_\omega}\delta\tilde{b}_\omega\right)\right)\left(\hat{f}_R - \hat{f}_R\mathbf{S}\left(\left(\left(\hat{R}_i^{L_i}\right)^{-1}\hat{R}_j^{L_i}\right)^{-1}\delta\theta_i\right) + \hat{f}_R\mathbf{S}\left(\delta\theta_j\right)\right)
\end{aligned}$$

where

$$\hat{f}_R = \hat{R}_{BC}^{-1}\left(\hat{R}_i^{L_i}\right)^{-1}\hat{R}_j^{L_i}$$

Further developing the expression we have:

$$\begin{aligned}
f_R \;&\approx\; \left(\mathbf{I} - \mathbf{S}\left(J_r(\theta_{BIAS})\frac{\partial R_j^i}{\partial b_\omega}\delta\tilde{b}_\omega\right)\right)\left(\hat{f}_R - \hat{f}_R\mathbf{S}\left(\left(\left(\hat{R}_i^{L_i}\right)^{-1}\hat{R}_j^{L_i}\right)^{-1}\delta\theta_i\right) + \hat{f}_R\mathbf{S}\left(\delta\theta_j\right)\right) = \\
&\approx \hat{f}_R - \hat{f}_R\mathbf{S}\left(\left(\left(\hat{R}_i^{L_i}\right)^{-1}\hat{R}_j^{L_i}\right)^{-1}\delta\theta_i\right) + \hat{f}_R\mathbf{S}\left(\delta\theta_j\right) - \mathbf{S}\left(J_r(\theta_{BC})\frac{\partial R_j^i}{\partial b_\omega}\delta\tilde{b}_\omega\right)\hat{f}_R = \\
&= \hat{f}_R\left[\mathbf{I} - \mathbf{S}\left(\left(\left(\hat{R}_i^{L_i}\right)^{-1}\hat{R}_j^{L_i}\right)^{-1}\delta\theta_i\right) + \mathbf{S}\left(\delta\theta_j\right) - \hat{f}_R^{-1}\mathbf{S}\left(J_r(\theta_{BIAS})\frac{\partial R_j^i}{\partial b_\omega}\delta\tilde{b}_\omega\right)\hat{f}_R\right] = \\
&= \hat{f}_R\left[\mathbf{I} - \mathbf{S}\left(\left(\left(\hat{R}_i^{L_i}\right)^{-1}\hat{R}_j^{L_i}\right)^{-1}\delta\theta_i\right) + \mathbf{S}\left(\delta\theta_j\right) - \mathbf{S}\left(\hat{f}_R^{-1}J_r(\theta_{BIAS})\frac{\partial R_j^i}{\partial b_\omega}\delta\tilde{b}_\omega\right)\right]
\end{aligned}$$

from which we obtain the desired Jacobians

$$
\begin{array}{rcllllllll}
J_R & = & [0 & -\left(\hat{R}_j^{L_i}\right)^{-1}\hat{R}_i^{L_i} & 0 & 0 & -\hat{f}_R^{-1}J_r(\theta_{BIAS})\frac{\partial R_j^i}{\partial b_\omega} & 0 & \mathbf{I} & 0] \\
\delta x & = & [\delta p_i & \delta\theta_i & \delta v_i & \delta\tilde{b}_{ai} & \delta\tilde{b}_{\omega i} & \delta p_j & \delta\theta_j & \delta v_j]^T
\end{array}
$$

2.3 Model 3: complete model with no body-sensor misalignment

Assumptions:

1. $R_{L_i}^{L_j} = \mathbf{I}$
2. known initial conditions $(g^{L_i}, p_0^{L_0}, v_0^{L_0}, R_0^{L_0})$

We first recall the expressions for the factors, approximating a first-order dependence of the preintegrated measurements on the biases:

$$
\begin{aligned}
f_p &= p_j^{L_j} - p_i^{L_i} - R_i^{L_i}\Delta p_{ij} - v_i^{L_i}\Delta t - \left(\frac{1}{2}g^{L_i} - \mathbf{S}(\omega_{iL_i}^{L_i})v_i^{L_i}\right)\Delta t^2 \\
f_v &= v_j^{L_j} - v_i^{L_i} - R_i^{L_i}\Delta v_{ij} - \left(g^{L_i} - 2\mathbf{S}(\omega_{iL_i}^{L_i})v_i^{L_i}\right)\Delta t \\
f_R &= \left(R_i^{L_i}exp\Delta\phi\right)^{-1}R_j^{L_j} = (exp\Delta\phi)^{-1}\left(R_i^{L_i}\right)^{-1}R_j^{L_j}, \quad with \quad \Delta\phi = log(R_j^i) - R_{L_i}^i\omega_{iL_i}^{L_i}\Delta t
\end{aligned}
$$

Then we proceed in the linearization of the factors.

2.3.1 Position

$$
\begin{aligned}
f_p &= p_j^{L_j} - p_i^{L_i} - R_i^{L_i}\Delta p_{ij} - v_i^{L_i}\Delta t - \left(\frac{1}{2}g^{L_i} - \mathbf{S}(\omega_{iL_i}^{L_i})v_i^{L_i}\right)\Delta t^2 \\
&= p_j^{L_j} - p_i^{L_i} - R_i^{L_i}\left(\Delta\hat{p}_{ij} + \frac{\partial\Delta p_{ij}}{\partial b_\omega}\tilde{b}_\omega + \frac{\partial\Delta p_{ij}}{\partial b_a}\tilde{b}_a\right) - v_i^{L_i}\Delta t \\
&\quad - \left(\frac{1}{2}g^{L_i} - \mathbf{S}(\omega_{iL_i}^{L_i})v_i^{L_i}\right)\Delta t^2
\end{aligned}
$$

Let us linearize the previous expression:

$$
\begin{aligned}
f_p &= \hat{p}_j^{L_j} + R_j^{L_i}\delta p_j - \hat{p}_i^{L_j} - R_i^{L_i}\delta p_i - R_i^{L_i}\left(\Delta\hat{p}_{ij} + \frac{\partial\Delta p_{ij}}{\partial b_\omega}\hat{\tilde{b}}_\omega + \frac{\partial\Delta p_{ij}}{\partial b_a}\hat{\tilde{b}}_a + \frac{\partial\Delta p_{ij}}{\partial b_\omega}\delta\tilde{b}_\omega + \frac{\partial\Delta p_{ij}}{\partial b_a}\delta\tilde{b}_a\right)\\
&\quad -\frac{1}{2}g^{L_i}\Delta t^2 + \left(\mathbf{S}(\omega_{iL_i}^{L_i})\Delta t^2 - \mathbf{I}\Delta t\right)\left(\hat{v}_i^{L_i} + \delta v_i\right)\\
&= \hat{p}_j^{L_j} + \hat{R}_j^{L_i}\delta R_j\delta p_j - \hat{p}_i^{L_j} - \hat{R}_i^{L_i}\delta R_i\delta p_i\\
&\quad -\hat{R}_i^{L_i}\delta R_i\left(\Delta\hat{p}_{ij} + \frac{\partial\Delta p_{ij}}{\partial b_\omega}\hat{\tilde{b}}_\omega + \frac{\partial\Delta p_{ij}}{\partial b_a}\hat{\tilde{b}}_a + \frac{\partial\Delta p_{ij}}{\partial b_\omega}\delta\tilde{b}_\omega + \frac{\partial\Delta p_{ij}}{\partial b_a}\delta\tilde{b}_a\right)\\
&\quad -\frac{1}{2}g^{L_i}\Delta t^2 + \left(\mathbf{S}(\omega_{iL_i}^{L_i})\Delta t^2 - \mathbf{I}\Delta t\right)\left(\hat{v}_i^{L_i} + \delta v_i\right)\\
&\approx \hat{p}_j^{L_j} + \hat{R}_j^{L_i}\left(\mathbf{I} + \mathbf{S}(\delta\theta_j)\right)\delta p_j - \hat{p}_i^{L_j} - \hat{R}_i^{L_i}\left(\mathbf{I} + \mathbf{S}(\delta\theta_i)\right)\delta p_i\\
&\quad -\hat{R}_i^{L_i}\left(\mathbf{I} + \mathbf{S}(\delta\theta_i)\right)\left(\Delta\hat{p}_{ij} + \frac{\partial\Delta p_{ij}}{\partial b_\omega}\hat{\tilde{b}}_\omega + \frac{\partial\Delta p_{ij}}{\partial b_a}\hat{\tilde{b}}_a + \frac{\partial\Delta p_{ij}}{\partial b_\omega}\delta\tilde{b}_\omega + \frac{\partial\Delta p_{ij}}{\partial b_a}\delta\tilde{b}_a\right)\\
&\quad -\frac{1}{2}g^{L_i}\Delta t^2 + \left(\mathbf{S}(\omega_{iL_i}^{L_i})\Delta t^2 - \mathbf{I}\Delta t\right)\left(\hat{v}_i^{L_i} + \delta v_i\right)
\end{aligned}
$$

Dropping second-order terms:

$$
\begin{aligned}
f_p &\approx \hat{p}_j^{L_j} + \hat{R}_j^{L_i}\delta p_j - \hat{p}_i^{L_j} - \hat{R}_i^{L_i}\delta p_i - \hat{R}_i^{L_i}\left(\Delta\hat{p}_{ij} + \frac{\partial\Delta p_{ij}}{\partial b_\omega}\hat{\tilde{b}}_\omega + \frac{\partial\Delta p_{ij}}{\partial b_a}\hat{\tilde{b}}_a + \frac{\partial\Delta p_{ij}}{\partial b_\omega}\delta\tilde{b}_\omega + \frac{\partial\Delta p_{ij}}{\partial b_a}\delta\tilde{b}_a\right)\\
&\quad -\hat{R}_i^{L_i}\mathbf{S}(\delta\theta_i)\left(\Delta\hat{p}_{ij} + \frac{\partial\Delta p_{ij}}{\partial b_\omega}\hat{\tilde{b}}_\omega + \frac{\partial\Delta p_{ij}}{\partial b_a}\hat{\tilde{b}}_a\right) - \frac{1}{2}g^{L_i}\Delta t^2 + \left(\mathbf{S}(\omega_{iL_i}^{L_i})\Delta t^2 - \mathbf{I}\Delta t\right)\left(\hat{v}_i^{L_i} + \delta v_i\right)\\
&= \hat{p}_j^{L_j} + \hat{R}_j^{L_i}\delta p_j - \hat{p}_i^{L_j} - \hat{R}_i^{L_i}\delta p_i - \hat{R}_i^{L_i}\left(\Delta\hat{p}_{ij} + \frac{\partial\Delta p_{ij}}{\partial b_\omega}\hat{\tilde{b}}_\omega + \frac{\partial\Delta p_{ij}}{\partial b_a}\hat{\tilde{b}}_a\right)\\
&\quad -\hat{R}_i^{L_i}\frac{\partial\Delta p_{ij}}{\partial b_\omega}\delta\tilde{b}_\omega - \hat{R}_i^{L_i}\frac{\partial\Delta p_{ij}}{\partial b_a}\delta\tilde{b}_a\\
&\quad +\hat{R}_i^{L_i}\mathbf{S}\left(\Delta\hat{p}_{ij} + \frac{\partial\Delta p_{ij}}{\partial b_\omega}\hat{\tilde{b}}_\omega + \frac{\partial\Delta p_{ij}}{\partial b_a}\hat{\tilde{b}}_a\right)\delta\theta_i - \frac{1}{2}g^{L_i}\Delta t^2 + \left(\mathbf{S}(\omega_{iL_i}^{L_i})\Delta t^2 - \mathbf{I}\Delta t\right)\left(\hat{v}_i^{L_i} + \delta v_i\right)
\end{aligned}
$$

Rearranging the terms we obtain:

$$
\begin{aligned}
f_p &\approx \left(\hat{p}_j^{L_i} - \hat{p}_i^{L_i} - \hat{R}_i^{L_i}\left(\Delta\hat{p}_{ij} + \frac{\partial\Delta p_{ij}}{\partial b_\omega}\hat{\tilde{b}}_\omega + \frac{\partial\Delta p_{ij}}{\partial b_a}\hat{\tilde{b}}_a\right) - \frac{1}{2}g^{L_i}\Delta t^2 + \left(\mathbf{S}(\omega_{iL_i}^{L_i})\Delta t^2 - \mathbf{I}\Delta t\right)\left(\hat{v}_i^{L_i}\right)\right)\\
&\quad +J_p\delta x
\end{aligned}
$$

with

$$
\begin{array}{rcccccccccc}
J_p &=& [-\hat{R}_i^{L_i} & J_{\theta_i}^p & \left[\mathbf{S}(\omega_{iL_i}^{L_i})\Delta t^2 - \mathbf{I}\Delta t\right] & -\hat{R}_i^{L_i}\frac{\partial\Delta p_{ij}}{\partial b_a} & -\hat{R}_i^{L_i}\frac{\partial\Delta p_{ij}}{\partial b_\omega} & \hat{R}_j^{L_i} & 0 & 0]\\
\delta x &=& [\delta p_i & \delta\theta_i & \delta v_i & \delta\tilde{b}_{ai} & \delta\tilde{b}_{\omega i} & \delta p_j & \delta\theta_j & \delta v_j]^T
\end{array}
$$

and $J^p_{\theta_i} = \hat{R}^{L_i}_i \mathbf{S}\left(\Delta\hat{p}_{ij} + \frac{\partial\Delta p_{ij}}{\partial b_\omega}\hat{\tilde{b}}_\omega + \frac{\partial\Delta p_{ij}}{\partial b_a}\hat{\tilde{b}}_a\right)$.

2.3.2 Velocity

$$\begin{aligned}
f_v &= v^{L_j}_j - v^{L_i}_i - R^{L_i}_i \Delta v_{ij} - \left(g^{L_i} - 2\mathbf{S}(\omega^{L_i}_{iL_i})v^{L_i}_i\right)\Delta t \\
&= v^{L_j}_j - v^{L_i}_i - R^{L_i}_i \left(\Delta\hat{v}_{ij} + \frac{\partial\Delta v_{ij}}{\partial b_\omega}\tilde{b}_\omega + \frac{\partial\Delta v_{ij}}{\partial b_a}\tilde{b}_a\right) - \left(g^{L_i} - 2\mathbf{S}(\omega^{L_i}_{iL_i})v^{L_i}_i\right)\Delta t
\end{aligned}$$

Let us linearize the previous expression:

$$\begin{aligned}
f_v &= \hat{v}^{L_j}_j + \delta v_j - \hat{v}^{L_j}_i - \delta v_i \\
&\quad -\hat{R}^{L_i}_i \delta R_i \left(\Delta\hat{v}_{ij} + \frac{\partial\Delta v_{ij}}{\partial b_\omega}\hat{\tilde{b}}_\omega + \frac{\partial\Delta v_{ij}}{\partial b_a}\hat{\tilde{b}}_a + \frac{\partial\Delta v_{ij}}{\partial b_\omega}\delta\tilde{b}_\omega + \frac{\partial\Delta v_{ij}}{\partial b_a}\delta\tilde{b}_a\right) \\
&\quad -g^{L_i}\Delta t + 2\mathbf{S}(\omega^{L_i}_{iL_i})\Delta t\left(\hat{v}^{L_j}_i + \delta v_i\right) \\
&\approx \hat{v}^{L_j}_j + \delta v_j - \hat{v}^{L_j}_i - \delta v_i \\
&\quad -\hat{R}^{L_i}_i \left(\mathbf{I} + \mathbf{S}(\delta\theta_i)\right)\left(\Delta\hat{v}_{ij} + \frac{\partial\Delta v_{ij}}{\partial b_\omega}\hat{\tilde{b}}_\omega + \frac{\partial\Delta v_{ij}}{\partial b_a}\hat{\tilde{b}}_a + \frac{\partial\Delta v_{ij}}{\partial b_\omega}\delta\tilde{b}_\omega + \frac{\partial\Delta v_{ij}}{\partial b_a}\delta\tilde{b}_a\right) \\
&\quad -g^{L_i}\Delta t + 2\mathbf{S}(\omega^{L_i}_{iL_i})\Delta t\left(\hat{v}^{L_j}_i + \delta v_i\right)
\end{aligned}$$

dropping second-order terms

$$\begin{aligned}
f_v &\approx \hat{v}^{L_j}_j + \delta v_j - \hat{v}^{L_j}_i - \delta v_i \\
&\quad -\hat{R}^{L_i}_i \left(\Delta\hat{v}_{ij} + \frac{\partial\Delta v_{ij}}{\partial b_\omega}\hat{\tilde{b}}_\omega + \frac{\partial\Delta v_{ij}}{\partial b_a}\hat{\tilde{b}}_a + \frac{\partial\Delta v_{ij}}{\partial b_\omega}\delta\tilde{b}_\omega + \frac{\partial\Delta v_{ij}}{\partial b_a}\delta\tilde{b}_a\right) \\
&\quad -\hat{R}^{L_i}_i \mathbf{S}(\delta\theta_i)\left(\Delta\hat{v}_{ij} + \frac{\partial\Delta v_{ij}}{\partial b_\omega}\hat{\tilde{b}}_\omega + \frac{\partial\Delta v_{ij}}{\partial b_a}\hat{\tilde{b}}_a\right) \\
&\quad -g^{L_i}\Delta t + 2\mathbf{S}(\omega^{L_i}_{iL_i})\Delta t\left(\hat{v}^{L_j}_i + \delta v_i\right) \\
&= \hat{v}^{L_j}_j + \delta v_j - \hat{v}^{L_j}_i - \delta v_i \\
&\quad -\hat{R}^{L_i}_i \left(\Delta\hat{v}_{ij} + \frac{\partial\Delta v_{ij}}{\partial b_\omega}\hat{\tilde{b}}_\omega + \frac{\partial\Delta v_{ij}}{\partial b_a}\hat{\tilde{b}}_a\right) \\
&\quad -\hat{R}^{L_i}_i \frac{\partial\Delta v_{ij}}{\partial b_\omega}\delta\tilde{b}_\omega - \hat{R}^{L_i}_i \frac{\partial\Delta v_{ij}}{\partial b_a}\delta\tilde{b}_a \\
&\quad +\hat{R}^{L_i}_i \mathbf{S}\left(\Delta\hat{v}_{ij} + \frac{\partial\Delta v_{ij}}{\partial b_\omega}\hat{\tilde{b}}_\omega + \frac{\partial\Delta v_{ij}}{\partial b_a}\hat{\tilde{b}}_a\right)\delta\theta_i \\
&\quad -g^{L_i}\Delta t + 2\mathbf{S}(\omega^{L_i}_{iL_i})\Delta t\left(\hat{v}^{L_j}_i + \delta v_i\right)
\end{aligned}$$

Rearranging the terms we obtain:

$$f_v \approx \left(\hat{v}_j^{L_j} - \hat{v}_i^{L_j} - \hat{R}_i^{L_i}\left(\Delta\hat{v}_{ij} + \frac{\partial \Delta v_{ij}}{\partial b_\omega}\hat{\tilde{b}}_\omega + \frac{\partial \Delta v_{ij}}{\partial b_a}\hat{\tilde{b}}_a\right) - g^{L_i}\Delta t + 2\mathbf{S}(\omega_{iL_i}^{L_i})\Delta t\left(\hat{v}_i^{L_j}\right)\right) + J_v \delta x$$

with

$$\begin{array}{rclcccccccc} J_p & = & [0 & J_{\theta_i}^v & \left[-\mathbf{I} + 2\mathbf{S}(\omega_{iL_i}^{L_i})\Delta t\right] & -\hat{R}_i^{L_i}\frac{\partial \Delta v_{ij}}{\partial b_a} & -\hat{R}_i^{L_i}\frac{\partial \Delta v_{ij}}{\partial b_\omega} & 0 & 0 & \mathbf{I}] \\ \delta x & = & [\delta p_i & \delta\theta_i & \delta v_i & \delta\tilde{b}_{ai} & \delta\tilde{b}_{\omega i} & \delta p_j & \delta\theta_j & \delta v_j]^T \end{array}$$

and $J_{\theta_i}^v = \hat{R}_i^{L_i}\mathbf{S}\left(\Delta\hat{v}_{ij} + \frac{\partial \Delta v_{ij}}{\partial b_\omega}\hat{\tilde{b}}_\omega + \frac{\partial \Delta v_{ij}}{\partial b_a}\hat{\tilde{b}}_a\right)$.

2.3.3 Rotation

$$\begin{aligned} f_R &= \left(exp\left(log(R_j^i) - R_{L_i}^i \omega_{iL_i}^{L_i}\Delta t\right)\right)^{-1}\left(R_i^{L_i}\right)^{-1} R_j^{L_j} \\ &= \left(exp\left(log\left(\hat{R}_j^i \exp\left(\frac{\partial R_j^i}{\partial b_\omega}\tilde{b}_\omega\right)\right) - R_{L_i}^i \omega_{iL_i}^{L_i}\Delta t\right)\right)^{-1}\left(R_i^{L_i}\right)^{-1} R_j^{L_j} \end{aligned}$$

Let us linearize the previous expression:

$$\begin{aligned} f_R = & \left(exp\left(log\left(\hat{R}_j^i \exp\left(\frac{\partial R_j^i}{\partial b_\omega}\hat{\tilde{b}}_\omega + \frac{\partial R_j^i}{\partial b_\omega}\delta\tilde{b}_\omega\right)\right) - \left(\hat{R}_i^{L_i}\delta R_i\right)^{-1}\omega_{iL_i}^{L_i}\Delta t\right)\right)^{-1} \\ & \left(\hat{R}_i^{L_i}\delta R_i\right)^{-1}\hat{R}_j^{L_i}\delta R_j \\ \approx & \left(exp\left(log\left(\hat{R}_{BC}\exp\left(J_r\left(\theta_{BIAS}\right)\left(\frac{\partial R_j^i}{\partial b_\omega}\delta\tilde{b}_\omega\right)\right)\right) - \left(\hat{R}_i^{L_i}\delta R_i\right)^{-1}\omega_{iL_i}^{L_i}\Delta t\right)\right)^{-1} \\ & \left(\hat{R}_i^{L_i}\delta R_i\right)^{-1}\hat{R}_j^{L_i}\delta R_j \end{aligned}$$

where we used (1) and we introduced:

$$\hat{R}_{BC} = \hat{R}_j^i \exp\left(\frac{\partial R_j^i}{\partial b_\omega}\hat{\tilde{b}}_\omega\right) \qquad \theta_{BIAS} = \log\left(\frac{\partial R_j^i}{\partial b_\omega}\hat{\tilde{b}}_\omega\right)$$

We continue the linearization by using (2):

$$\begin{aligned}
f_R &\approx \left(exp\left(log\left(\hat{R}_{BC}\exp\left(J_r\left(\theta_{BIAS}\right)\left(\frac{\partial R_j^i}{\partial b_\omega}\delta\tilde{b}_\omega\right)\right)\right)\right) - \left(\hat{R}_i^{L_i}\delta R_i\right)^{-1}\omega_{iL_i}^{L_i}\Delta t\right)\right)^{-1} \\
&\quad \left(\hat{R}_i^{L_i}\delta R_i\right)^{-1}\hat{R}_j^{L_i}\delta R_j \approx \\
&\approx \left(exp\left(\theta_{BC} + J_r^{-1}\left(\theta_{BC}\right)J_r\left(\theta_{BIAS}\right)\left(\frac{\partial R_j^i}{\partial b_\omega}\delta\tilde{b}_\omega\right) - \left(\hat{R}_i^{L_i}\delta R_i\right)^{-1}\omega_{iL_i}^{L_i}\Delta t\right)\right)^{-1} \\
&\quad \left(\hat{R}_i^{L_i}\delta R_i\right)^{-1}\hat{R}_j^{L_i}\delta R_j
\end{aligned}$$

with (BC = bias corrected measurement)

$$\theta_{BC} = \log\left(\hat{R}_{BC}\right)$$

Taking small angle approximations

$$\begin{aligned}
f_R &\approx \left(exp\left(\theta_{BC} + J_r^{-1}\left(\theta_{BC}\right)J_r\left(\theta_{BIAS}\right)\left(\frac{\partial R_j^i}{\partial b_\omega}\delta\tilde{b}_\omega\right) - \left(\mathbf{I} - \mathbf{S}(\delta\theta_i)\right)\left(\hat{R}_i^{L_i}\right)^{-1}\omega_{iL_i}^{L_i}\Delta t\right)\right)^{-1} \\
&\quad \left(\mathbf{I} - \mathbf{S}(\delta\theta_i)\right)\left(\hat{R}_i^{L_i}\right)^{-1}\hat{R}_j^{L_i}\left(\mathbf{I} + \mathbf{S}(\delta\theta_j)\right) \\
&= \left(exp\left(\theta_{BC} + J_r^{-1}\left(\theta_{BC}\right)J_r\left(\theta_{BIAS}\right)\left(\frac{\partial R_j^i}{\partial b_\omega}\delta\tilde{b}_\omega\right) - \left(\hat{R}_i^{L_i}\right)^{-1}\omega_{iL_i}^{L_i}\Delta t + \mathbf{S}(\delta\theta_i)\left(\hat{R}_i^{L_i}\right)^{-1}\omega_{iL_i}^{L_i}\Delta t\right)\right)^{-1} \\
&\quad \left(\mathbf{I} - \mathbf{S}(\delta\theta_i)\right)\left(\hat{R}_i^{L_i}\right)^{-1}\hat{R}_j^{L_i}\left(\mathbf{I} + \mathbf{S}(\delta\theta_j)\right) \\
&= \left(exp\left(\theta_{BC} + J_r^{-1}\left(\theta_{BC}\right)J_r\left(\theta_{BIAS}\right)\left(\frac{\partial R_j^i}{\partial b_\omega}\delta\tilde{b}_\omega\right) - \left(\hat{R}_i^{L_i}\right)^{-1}\omega_{iL_i}^{L_i}\Delta t - \mathbf{S}\left(\left(\left(\hat{R}_i^{L_i}\right)^{-1}\omega_{iL_i}^{L_i}\Delta t\right)\delta\theta_i\right)\right)\right)^{-1} \\
&\quad \left(\mathbf{I} - \mathbf{S}(\delta\theta_i)\right)\left(\hat{R}_i^{L_i}\right)^{-1}\hat{R}_j^{L_i}\left(\mathbf{I} + \mathbf{S}(\delta\theta_j)\right)
\end{aligned}$$

Applying again(1) to the exponential map

$$
\begin{aligned}
f_R \quad \approx \quad & \left(exp\left(\theta_{BCC}\right)exp\left(J_r\left(\theta_{BCC}\right)J_r^{-1}\left(\theta_{BC}\right)J_r\left(\theta_{BIAS}\right)\left(\frac{\partial R_j^i}{\partial b_\omega}\delta\tilde{b}_\omega\right)-J_r\left(\theta_{BCC}\right)\mathbf{S}\left(\left(\left(\hat{R}_i^{L_i}\right)^{-1}\omega_{iL_i}^{L_i}\Delta t\right)\delta\theta_i\right)\right)\right)^{-1} \\
& \left(\mathbf{I}-\mathbf{S}(\delta\theta_i)\right)\left(\hat{R}_i^{L_i}\right)^{-1}\hat{R}_j^{L_i}\left(\mathbf{I}+\mathbf{S}(\delta\theta_j)\right) \\
= \quad & \left(exp\left(\theta_{BCC}\right)\left[\mathbf{I}+\mathbf{S}\left(J_r\left(\theta_{BCC}\right)J_r^{-1}\left(\theta_{BC}\right)J_r\left(\theta_{BIAS}\right)\left(\frac{\partial R_j^i}{\partial b_\omega}\delta\tilde{b}_\omega\right)-J_r\left(\theta_{BCC}\right)\mathbf{S}\left(\left(\left(\hat{R}_i^{L_i}\right)^{-1}\omega_{iL_i}^{L_i}\Delta t\right)\delta\theta_i\right)\right)\right]\right)^{-1} \\
& \left(\mathbf{I}-\mathbf{S}(\delta\theta_i)\right)\left(\hat{R}_i^{L_i}\right)^{-1}\hat{R}_j^{L_i}\left(\mathbf{I}+\mathbf{S}(\delta\theta_j)\right) \qquad (small\ angle\ approximation)
\end{aligned}
$$

with (BCC = bias and Coriolis corrected measurement)

$$
\theta_{BCC} = exp\left(\theta_{BC}-\left(\hat{R}_i^{L_i}\right)^{-1}\omega_{iL_i}^{L_i}\Delta t\right)
$$

Let us define

$$
\hat{R}_{BCC} = \exp\left(\theta_{BCC}\right) \qquad\qquad \hat{f}_R = \left(\hat{R}_{BCC}\right)^{-1}\left(\hat{R}_i^{L_i}\right)^{-1}\hat{R}_j^{L_i}
$$

Then we continue our derivation dropping higher order terms

$$
\begin{aligned}
f_R \approx\; & \left[\mathbf{I}-\mathbf{S}\left(J_r\left(\theta_{BCC}\right)J_r^{-1}\left(\theta_{BC}\right)J_r\left(\theta_{BIAS}\right)\left(\frac{\partial R_j^i}{\partial b_\omega}\delta\tilde{b}_\omega\right)-J_r\left(\theta_{BCC}\right)\mathbf{S}\left(\left(\hat{R}_i^{L_i}\right)^{-1}\omega_{iL_i}^{L_i}\Delta t\right)\delta\theta_i\right)\right] \\
& \left(\hat{R}_{BCC}\right)^{-1}\left[\left(\hat{R}_i^{L_i}\right)^{-1}\hat{R}_j^{L_i}-\mathbf{S}(\delta\theta_i)\left(\hat{R}_i^{L_i}\right)^{-1}\hat{R}_j^{L_i}+\left(\hat{R}_i^{L_i}\right)^{-1}\hat{R}_j^{L_i}\mathbf{S}(\delta\theta_j)\right] \\
=\; & \left[\mathbf{I}-\mathbf{S}\left(J_r\left(\theta_{BCC}\right)J_r^{-1}\left(\theta_{BC}\right)J_r\left(\theta_{BIAS}\right)\left(\frac{\partial R_j^i}{\partial b_\omega}\delta\tilde{b}_\omega\right)-J_r\left(\theta_{BCC}\right)\mathbf{S}\left(\left(\hat{R}_i^{L_i}\right)^{-1}\omega_{iL_i}^{L_i}\Delta t\right)\delta\theta_i\right)\right] \\
& \left(\hat{R}_{BCC}\right)^{-1}\left[\left(\hat{R}_i^{L_i}\right)^{-1}\hat{R}_j^{L_i}-\left(\hat{R}_i^{L_i}\right)^{-1}\hat{R}_j^{L_i}\left(\left(\hat{R}_i^{L_i}\right)^{-1}\hat{R}_j^{L_i}\right)^{-1}\mathbf{S}(\delta\theta_i)\left(\hat{R}_i^{L_i}\right)^{-1}\hat{R}_j^{L_i}+\left(\hat{R}_i^{L_i}\right)^{-1}\hat{R}_j^{L_i}\mathbf{S}(\delta\theta_j)\right] \\
=\; & \left[\mathbf{I}-\mathbf{S}\left(J_r\left(\theta_{BCC}\right)J_r^{-1}\left(\theta_{BC}\right)J_r\left(\theta_{BIAS}\right)\left(\frac{\partial R_j^i}{\partial b_\omega}\delta\tilde{b}_\omega\right)-J_r\left(\theta_{BCC}\right)\mathbf{S}\left(\left(\hat{R}_i^{L_i}\right)^{-1}\omega_{iL_i}^{L_i}\Delta t\right)\delta\theta_i\right)\right] \\
& \left(\hat{R}_{BCC}\right)^{-1}\left[\left(\hat{R}_i^{L_i}\right)^{-1}\hat{R}_j^{L_i}-\left(\hat{R}_i^{L_i}\right)^{-1}\hat{R}_j^{L_i}\mathbf{S}\left(\left(\left(\hat{R}_i^{L_i}\right)^{-1}\hat{R}_j^{L_i}\right)^{-1}\delta\theta_i\right)+\left(\hat{R}_i^{L_i}\right)^{-1}\hat{R}_j^{L_i}\mathbf{S}(\delta\theta_j)\right] \\
=\; & \left[\mathbf{I}-\mathbf{S}\left(J_r\left(\theta_{BCC}\right)J_r^{-1}\left(\theta_{BC}\right)J_r\left(\theta_{BIAS}\right)\left(\frac{\partial R_j^i}{\partial b_\omega}\delta\tilde{b}_\omega\right)-J_r\left(\theta_{BCC}\right)\mathbf{S}\left(\left(\hat{R}_i^{L_i}\right)^{-1}\omega_{iL_i}^{L_i}\Delta t\right)\delta\theta_i\right)\right] \\
& \hat{f}_R\left[\mathbf{I}-\mathbf{S}\left(\left(\left(\hat{R}_i^{L_i}\right)^{-1}\hat{R}_j^{L_i}\right)^{-1}\delta\theta_i\right)+\mathbf{S}(\delta\theta_j)\right] \\
\approx\; & \hat{f}_R-\mathbf{S}\left(J_r\left(\theta_{BCC}\right)J_r^{-1}\left(\theta_{BC}\right)J_r\left(\theta_{BIAS}\right)\left(\frac{\partial R_j^i}{\partial b_\omega}\delta\tilde{b}_\omega\right)-J_r\left(\theta_{BCC}\right)\mathbf{S}\left(\left(\hat{R}_i^{L_i}\right)^{-1}\omega_{iL_i}^{L_i}\Delta t\right)\delta\theta_i\right)\hat{f}_R \\
& -\hat{f}_R\left[\mathbf{S}\left(\left(\left(\hat{R}_i^{L_i}\right)^{-1}\hat{R}_j^{L_i}\right)^{-1}\delta\theta_i\right)+\mathbf{S}(\delta\theta_j)\right]= \\
=\; & \hat{f}_R-\hat{f}_R\mathbf{S}\left[\hat{f}_R^{-1}\left(J_r\left(\theta_{BCC}\right)J_r^{-1}\left(\theta_{BC}\right)J_r\left(\theta_{BIAS}\right)\left(\frac{\partial R_j^i}{\partial b_\omega}\delta\tilde{b}_\omega\right)-J_r\left(\theta_{BCC}\right)\mathbf{S}\left(\left(\hat{R}_i^{L_i}\right)^{-1}\omega_{iL_i}^{L_i}\Delta t\right)\delta\theta_i\right)\right] \\
& \hat{f}_R\left[-\mathbf{S}\left(\left(\left(\hat{R}_i^{L_i}\right)^{-1}\hat{R}_j^{L_i}\right)^{-1}\delta\theta_i\right)+\mathbf{S}(\delta\theta_j)\right]=
\end{aligned}
$$

from which we obtain the desired Jacobians

$$
\begin{aligned}
J_R &= [0 \quad J_{\theta_i}^R \quad 0 \quad 0 \quad J_\omega^R \quad 0 \quad \mathbf{I} \quad 0] \\
\delta x &= [\delta p_i \quad \delta\theta_i \quad \delta v_i \quad \delta\tilde{b}_{ai} \quad \delta\tilde{b}_{\omega i} \quad \delta p_j \quad \delta\theta_j \quad \delta v_j]^T
\end{aligned}
$$

with

$$
\begin{aligned}
J_{\theta_i}^R &= -\left(\left(\hat{R}_i^{L_i}\right)^{-1}\hat{R}_j^{L_i}\right)^{-1}+\hat{f}_R^{-1}\left(J_r\left(\theta_{BCC}\right)\mathbf{S}\left(\left(\hat{R}_i^{L_i}\right)^{-1}\omega_{iL_i}^{L_i}\Delta t\right)\right) \\
&= -\left(\hat{R}_j^{L_i}\right)^{-1}\hat{R}_i^{L_i}+\hat{f}_R^{-1}\left(J_r\left(\theta_{BCC}\right)\mathbf{S}\left(\left(\hat{R}_i^{L_i}\right)^{-1}\omega_{iL_i}^{L_i}\Delta t\right)\right) \\
J_\omega^R &= -\hat{f}_R^{-1}\left(J_r\left(\theta_{BCC}\right)J_r^{-1}\left(\theta_{BC}\right)J_r\left(\theta_{BIAS}\right)\left(\frac{\partial R_j^i}{\partial b_\omega}\right)\right)
\end{aligned}
$$